Sum with tangents in square.

https://www.linkedin.com/feed/update/urn:li:activity:6880007990989197312

Evaluate $\sum\limits_{k=1}^{1010} \tan^2 \dfrac{k\pi}{2022}$

Solution by Arkady Alt, **San Jose**, **California**, **USA**.

Since $\sum\limits_{k=1}^{n-1} \tan^2 \dfrac{k\pi}{2n} = \sum\limits_{k=1}^{n-1}\left(\dfrac{1}{\cos^2 \frac{k\pi}{2n}} - 1\right) = \sum\limits_{k=1}^{n-1} \dfrac{1}{\cos^2 \frac{k\pi}{2n}} - (n-1)$

remains calculate $\sum\limits_{k=1}^{n-1} \dfrac{1}{\cos^2 \frac{k\pi}{2n}}$.

First note, that for any polynomial $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ $(a_0 \neq 0)$

with non-zero roots $x_1, x_2, \ldots, x_n$ holds

(**1**) $\sum\limits_{i=1}^{n} \dfrac{1}{x_i} = -\dfrac{P'(0)}{P(0)}$.

Indeed, since $P(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n)$ then

$$\sum_{i=1}^{n} \frac{1}{x-x_i} = \sum_{i=1}^{n}(\ln(x-x_i))' = \left(\sum_{i=1}^{n}\ln(x-x_i)\right)' = \left(\ln\frac{P(x)}{a_0}\right)' = \frac{P'(x)}{P(x)}.$$

Hence, $\sum\limits_{i=1}^{n} \dfrac{1}{x_i} = -\dfrac{P'(0)}{P(0)}$. ■

For any real θ let $u_n(\theta) := \dfrac{\sin((n+1)\theta)}{\sin\theta}, n \in \mathbb{N} \cup \{0\}$. Since $\sin((n+2)\theta) + \sin n\theta =$

$2\cos\theta \cdot \sin((n+1)\theta)$ then u_n can be defined by recurrence

(**2**) $u_{n+1}(\theta) = 2\cos\theta \cdot u_n(\theta) - u_{n-1}(\theta), n \in \mathbb{N}$ and $u_0(\theta) = 1, u_1(\theta) = 2\cos\theta$.

If we replace $\cos\theta$ in (2) with real x we obtain recurrence

(**3**) $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), n \in \mathbb{N}$ and initial condition $U_0 = 1, U_1 = 2x$

which define polynomials $U_n(x), n \in \mathbb{N} \cup \{0\}$ (Chebishev's polynomials of the 2nd kind).

For example $U_2(x) = 4x^2 - 1, U_3(x) = 8x^3 - 4x$.

Since $U_n(\cos\theta) = u_n(\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$ then for $\theta \in (0,\pi)$ we have

$U_n(\cos\theta) = 0 \Leftrightarrow \dfrac{\sin(n+1)\theta}{\sin\theta} = 0 \Leftrightarrow \theta = \dfrac{k\pi}{n+1}, k = 1,2,\ldots,n.$

Since $\deg U_n(x) = n, x = \cos\theta$ and coefficient for x^n equal* 2^n then $U_n(x) = 0 \Leftrightarrow$

$x = \cos\dfrac{k\pi}{n+1}, k = 1,2,\ldots,n$ and therefore

$$U_n(x) = 2^n\left(x - \cos\frac{\pi}{n+1}\right)\left(x - \cos\frac{2\pi}{n+1}\right)\ldots\left(x - \cos\frac{n\pi}{n+1}\right).$$

* If α_n is coefficient for x^n in $U_n(x)$ then α_n satisfies to recurrence $\alpha_{n+1} = 2\alpha_n, \alpha_0 = 1$).

In particularly $U_{2n-1}(x) = 2^{2n-1}\prod\limits_{k=1}^{2n-1}\left(x - \cos\dfrac{k\pi}{2n}\right) =$

$$2^{2n+1}x\prod_{k=1}^{n-1}\left(x - \cos\frac{k\pi}{2n}\right)\prod_{k=1}^{n-1}\left(x - \cos\frac{(2n-k)\pi}{2n}\right) = 2^{2n-1}x\prod_{k=1}^{n-1}\left(x^2 - \cos^2\frac{k\pi}{2n}\right).$$

Let $P_n(x) := \dfrac{U_{2n-1}(\sqrt{x})}{2\sqrt{x}}$ then $P_n(x) = 4^{n-1}\prod\limits_{k=1}^{n-1}\left(x - \cos^2\dfrac{k\pi}{2n}\right)$.

Note that $U_{2n-1}(x)$ can be directly defined by recurrence

$U_{2n+1}(x) = 2(2x^2-1)U_{2n-1}(x) - U_{2n-3}(x), n \in \mathbb{N}$ with $U_{-1}(x) = 0, U_1(x) = 2x$.
Since $U_{2n-1}(x)$ divisible by $2x$ then polynomial $P_n(x)$ satisfy to the recurrence
(**4**) $P_{n+1}(x) = 2(2x-1)P_n(x) - P_{n-1}(x), n \in \mathbb{N}$ with $P_0(x) = 0, P_1(x) = 1$.

Thus, applying (1) to polynomial $P_n(x)$ we have

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = -\frac{P_n'(0)}{P_n(0)}.$$

In particularly, from (4) follows recurrence
(5) $P_{n+1}(0) + 2P_n(0) + P_{n-1}(0) = 0, n \in \mathbb{N}$ with $P_0(0) = 0, P_1(0) = 1$.
Let $b_n := \frac{P_n(0)}{(-1)^n}$ then (5) can be rewritten as
(**6**) $b_{n+1} - 2b_n + b_{n-1} = 0, n \in \mathbb{N}$ with $b_0 = 0, b_1 = -1$.
Since $b_{n+1} - b_n = b_n - b_{n-1}, n \in \mathbb{N}$ then $b_n - b_{n-1} = -1, n \in \mathbb{N}$ and, therefore,

$$\sum_{k=1}^{n}(b_k - b_{k-1}) = -n \Leftrightarrow b_n - b_0 = -n \Leftrightarrow b_n = -n.$$

From the other hand, since $P_{n+1}'(x) = 2(2x-1)P_n'(x) + 4P_n(x) - P_{n-1}'(x), n \in \mathbb{N}$
with $P_0'(x) = 0, P_1'(x) = 0$, then
(**7**) $P_{n+1}'(0) + 2P_n'(0) + P_{n-1}'(0) = 4P_n(0), n \in \mathbb{N}$ with $P_0'(0) = 0, P_1'(0) = 0$.
Let $a_n := \frac{P_n'(0)}{(-1)^n}$ then $\frac{P_n(0)}{(-1)^{n+1}} = -b_n = n$ and (**7**) can be rewritten as
(**8**) $a_{n+1} - 2a_n + a_{n-1} = 4n, n \in \mathbb{N}$ with $a_0 = a_1 = 0$.
Since sequence $\left(\frac{2n(n^2-1)}{3}\right)$ is particular solution of nonhomogeneous recurrence (8) then $a_n = \frac{2n(n^2-1)}{3} + \alpha n + \beta$ where $\alpha = \beta = 0$ because $a_0 = a_1 = 0$.
Thus $a_n = \frac{2n(n^2-1)}{3}, n \in \mathbb{N}$ and, therefore,

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = -\frac{P_n'(0)}{P_n(0)} = \frac{\frac{P_n'(0)}{(-1)^n}}{\frac{P_n(0)}{(-1)^{n+1}}} = \frac{a_n}{n} = \frac{2(n^2-1)}{3}.$$

Hence, $\sum_{k=1}^{n-1} \tan^2 \frac{k\pi}{2n} = \frac{2(n^2-1)}{3} - (n-1) = \frac{(2n-1)(n-1)}{3}$.
In particular for $n = 1010$ we obtain

$$\sum_{k=1}^{1010} \tan^2 \frac{k\pi}{2022} = \frac{(2 \cdot 1011 - 1)(1011-1)}{3} = 6.804 \times 10^5.$$